Problem 1.43

(a) Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2\phi)\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of $\mathbf{v}$.

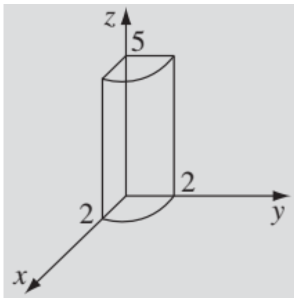


Fig. 1.43

Solution

In cylindrical coordinates (s, ϕ, z) the curl of a vector function is

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}},$$

and the divergence of a vector function is

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

For $\mathbf{v} = s(2 + \sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}$, the curl is

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left[\frac{1}{s}\frac{\partial}{\partial \phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right]\hat{\mathbf{s}} + \left[\frac{\partial}{\partial z}(2s + s\sin^2\phi) - \frac{\partial}{\partial s}(3z)\right]\hat{\boldsymbol{\phi}} \\
&\qquad + \frac{1}{s}\left\{\frac{\partial}{\partial s}[s(s\sin\phi\cos\phi)] - \frac{\partial}{\partial \phi}(2s + s\sin^2\phi)\right\}\hat{\mathbf{z}} \\
&= \left[\frac{1}{s}(0) - (0)\right]\hat{\mathbf{s}} + [(0) - (0)]\,\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[(2s\sin\phi\cos\phi) - (2s\sin\phi\cos\phi)\right]\hat{\mathbf{z}} \\
&= 0\,\hat{\mathbf{s}} + 0\,\hat{\boldsymbol{\phi}} + 0\,\hat{\mathbf{z}} \\
&= \mathbf{0},
\end{aligned}$$

and the divergence is

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}[s(2s+s\sin^2\phi)] + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}\frac{\partial}{\partial s}[s^2(2+\sin^2\phi)] + \frac{1}{s}\frac{\partial}{\partial\phi}\left(\frac{s}{2}\sin 2\phi\right) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}[2s(2+\sin^2\phi)] + \frac{1}{s}(s\cos 2\phi) + (3)\\ &= 4 + 2\sin^2\phi + \cos 2\phi + 3\\ &= 4 + 2\sin^2\phi + (1 - 2\sin^2\phi) + 3\\ &= 8.\end{aligned}$$

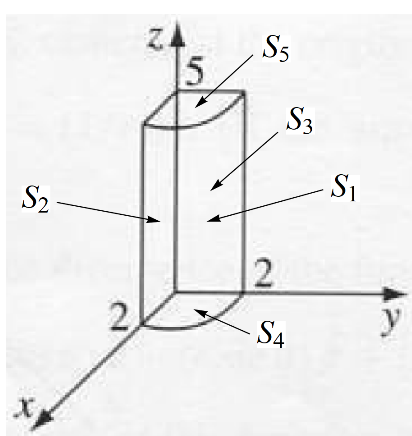
The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla\cdot\mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla\cdot\mathbf{v}\,dV = \oiint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S}$$

If D is the quarter-cylinder shown in Fig. 1.43, then the left side evaluates to

$$\begin{aligned}\iiint_D \nabla\cdot\mathbf{v}\,dV &= \int_0^5\int_0^{\pi/2}\int_0^2 (\nabla\cdot\mathbf{v})(s\,ds\,d\phi\,dz)\\ &= \int_0^5\int_0^{\pi/2}\int_0^2 (8)(s\,ds\,d\phi\,dz)\\ &= 8\left(\int_0^5 dz\right)\left(\int_0^{\pi/2} d\phi\right)\left(\int_0^2 s\,ds\right)\\ &= 8(5)\left(\frac{\pi}{2}\right)(2)\\ &= 40\pi,\end{aligned}$$

and with the sides labelled as shown below,



the right side evaluates to

$$\oiint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{v}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{v}\cdot d\mathbf{S} + \iint_{S_3} \mathbf{v}\cdot d\mathbf{S} + \iint_{S_4} \mathbf{v}\cdot d\mathbf{S} + \iint_{S_5} \mathbf{v}\cdot d\mathbf{S}$$

$$= \int_0^5 \int_0^{\pi/2} [s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}]\Big|_{s=2} \cdot (\hat{\mathbf{s}}2\,d\phi\,dz)$$

$$+ \int_0^5 \int_0^2 [s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}]\Big|_{\phi=0} \cdot (-\hat{\boldsymbol{\phi}}\,dx\,dz)$$

$$+ \int_0^5 \int_0^2 [s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}]\Big|_{\phi=\pi/2} \cdot (\hat{\boldsymbol{\phi}}\,dy\,dz)$$

$$+ \int_0^{\pi/2} \int_0^2 [s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}]\Big|_{z=0} \cdot (-\hat{\mathbf{z}}s\,ds\,d\phi)$$

$$+ \int_0^{\pi/2} \int_0^2 [s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}]\Big|_{z=5} \cdot (\hat{\mathbf{z}}s\,ds\,d\phi)$$

$$= \int_0^5 \int_0^{\pi/2} [2(2+\sin^2\phi)\,\hat{\mathbf{s}} + 2\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}] \cdot (\hat{\mathbf{s}}2\,d\phi\,dz)$$

$$+ \int_0^5 \int_0^2 [s(2+0)\,\hat{\mathbf{s}} + s(0)(1)\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}] \cdot (-\hat{\boldsymbol{\phi}}\,dx\,dz)$$

$$+ \int_0^5 \int_0^2 [s(2+1)\,\hat{\mathbf{s}} + s(1)(0)\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}] \cdot (\hat{\boldsymbol{\phi}}\,dy\,dz)$$

$$+ \int_0^{\pi/2} \int_0^2 [s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3(0)\,\hat{\mathbf{z}}] \cdot (-\hat{\mathbf{z}}s\,ds\,d\phi)$$

$$+ \int_0^{\pi/2} \int_0^2 [s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3(5)\,\hat{\mathbf{z}}] \cdot (\hat{\mathbf{z}}s\,ds\,d\phi).$$

Only the first and last integrals are nonzero.

$$\begin{aligned}
\oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S} &= \int_0^5 \int_0^{\pi/2} 4(2 + \sin^2 \phi) \, d\phi \, dz + \int_0^{\pi/2} \int_0^2 (15s) \, ds \, d\phi \\
&= 4 \left(\int_0^5 dz \right) \left[\int_0^{\pi/2} (2 + \sin^2 \phi) \, d\phi \right] + 15 \left(\int_0^{\pi/2} d\phi \right) \left(\int_0^2 s \, ds \right) \\
&= 4 \left(\int_0^5 dz \right) \left[\int_0^{\pi/2} \left(2 + \frac{1}{2} - \frac{1}{2} \cos 2\phi \right) d\phi \right] + 15 \left(\int_0^{\pi/2} d\phi \right) \left(\int_0^2 s \, ds \right) \\
&= 4(5) \left(\frac{5\pi}{4} \right) + 15 \left(\frac{\pi}{2} \right) (2) \\
&= 40\pi
\end{aligned}$$